# Convex Analysis Background

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**Def:** $f : \mathbb{R}^n \to [-\infty, \infty]$ is differentiable at $\bar{x} \in \mathbb{R}^n$ if $f(\bar{x}) \in \mathbb{R}$ and there exists a linear map $f'(\bar{x}) : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(\bar{x} + h) - f(\bar{x}) - f'(\bar{x})h}{\|h\|} = 0 \tag{1}$$

**Facts:** If $f$ is differentiable at $\bar{x}$, then

1) $f'(\bar{x})$ satisfying (**??**) is unique
2) $\bar{x} \in \mathrm{int}\,(\mathrm{dom}\, f)$ where

$$\mathrm{dom}\, f := \{x \in \mathbb{R}^n : f(x) < \infty\}$$

**Def:** Given function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, define the epigraph and strict epigraph, respectively, as

$$\operatorname{epi} f := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\}$$
$$\operatorname{epi}_s f := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : f(x) < t\}$$

**Def:** $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be **convex** if its epigraph $\operatorname{epi} f$ is a convex set. The set of all such functions is denoted by $\text{E-Conv}(\mathbb{R}^n)$

## Proposition

*The following conditions about* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ *are equivalent:*

a) $\operatorname{epi} f$ *is a convex set* (*i.e.,* $f \in \text{E-Conv}(\mathbb{R}^n)$)

b) $\operatorname{epi}_s f$ *is a convex set*

c) *for every* $x_0, x_1 \in \operatorname{dom} f$ *and* $\alpha \in (0, 1)$, *we have*

$$f(\alpha x_0 + (1-\alpha)x_1) \leq \alpha f(x_0) + (1-\alpha)f(x_1)$$

**Notation:** Let

$$\Delta_k := \left\{ (\alpha_0, \ldots, \alpha_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^{k} \alpha_i = 1,\ \alpha_i \geq 0 \right\},$$

**Proposition**

*Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be given. Then, $f \in \mathrm{E\text{-}Conv}(\mathbb{R}^n)$ iff*

$$f(\alpha_0 x_0 + \ldots + \alpha_k x_k) \leq \alpha_0 f(x_0) + \ldots + \alpha_k f(x_k),$$

*for every $x_0, \ldots, x_k \in \mathrm{dom}\, f$ and $(\alpha_0, \ldots, \alpha_k) \in \Delta_k$*

*In particular, for every $x_0, \ldots, x_k \in \mathrm{dom}\, f$ and $x \in \mathrm{co}\,\{x_0, \ldots, x_k\}$, there holds*

$$f(x) \leq \max\{f(x_i) : i = 0, \ldots, k\}$$

**Def:** $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is **proper** if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathbb{R}^n$. The set of all proper convex functions is denoted by $\mathrm{Conv}(\mathbb{R}^n)$

**Def:** $f \in \mathrm{Conv}(\mathbb{R}^n)$ is **closed** if its epigraph is closed. The set of all proper closed convex functions is denoted by $\overline{\mathrm{Conv}}\,(\mathbb{R}^n)$

**Definition**

The indicator function $\delta_C : \mathbb{R}^n \to [0, \infty]$ of a set $C \subset \mathbb{R}^n$ is

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise} \end{cases}$$

**Obs:** $\delta_C$ is convex (resp., proper convex) $\Leftrightarrow$ $C$ is convex (resp., nonempty convex)

**Definition**

The support function $\sigma_C : \mathbb{R}^n \to [-\infty, \infty]$ of $C$ is

$$\sigma_C(x) := \sup\{\langle x, c\rangle : c \in C\}$$

If $C = \emptyset$ then $\sigma_C = -\infty$; otherwise, $\sigma_C \in \overline{\mathrm{Conv}}\,(\mathbb{R}^n)$

## Definition

$f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is **strictly convex** if it is proper and, for every $x_0 \neq x_1 \in \operatorname{dom} f$ and $\alpha \in (0, 1)$,

$$f(\alpha x_0 + (1-\alpha)x_1) < \alpha f(x_0) + (1-\alpha) f(x_1)$$

**Def:** For $\mu \geq 0$, define

$$\mathrm{Conv}_{\mu}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to (\infty, \infty] : f - \mu\|\cdot\|^2/2 \in \mathrm{Conv}(\mathbb{R}^n)\}$$
$$\overline{\mathrm{Conv}}_{\mu}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to (\infty, \infty] : f - \mu\|\cdot\|^2/2 \in \overline{\mathrm{Conv}}(\mathbb{R}^n)\}$$

Clearly,

$$\mathrm{Conv}_0(\mathbb{R}^n) = \mathrm{Conv}(\mathbb{R}^n) \quad \overline{\mathrm{Conv}}_0(\mathbb{R}^n) = \overline{\mathrm{Conv}}(\mathbb{R}^n)$$

**Def:** Function $f \in \mathrm{Conv}_{\mu}(\mathbb{R}^n)$ is called a $\mu$-strongly convex function if $\mu > 0$, or simply, a $\mu$-convex function if $\mu \geq 0$

**Def:** Let $f : \mathbb{R}^n \to (-\infty, \infty]$ and nonempty convex set $C \subset \operatorname{dom} f$ be given. $f$ is said to be **strictly convex on** $C$ if $f + \delta_C$ is strictly convex

**Proposition**

*If $f : \mathbb{R}^n \to (-\infty, \infty]$ is strictly convex on a nonempty convex set $C$, then*

$$\min\{f(x) : x \in C\}$$

*has at most one global minimum*

**Obs:** Previous result does not guarantee the existence of a global minimum but simply that there exists at most one (if any).

**Def:** For nonempty convex set $C$ and $\mu \geq 0$, define

$$\begin{aligned}
\mathrm{Conv}(C) &:= \{f : C \subset \operatorname{dom} f,\ f + \delta_C \in \mathrm{Conv}(\mathbb{R}^n)\} \\
\overline{\mathrm{Conv}}(C) &:= \{f : C \subset \operatorname{dom} f,\ f + \delta_C \in \overline{\mathrm{Conv}}(\mathbb{R}^n)\} \\
\mathrm{Conv}_\mu(C) &:= \{f : C \subset \operatorname{dom} f,\ f + \delta_C \in \mathrm{Conv}_\mu(\mathbb{R}^n)\} \\
\overline{\mathrm{Conv}}_\mu(C) &:= \{f : C \subset \operatorname{dom} f,\ f + \delta_C \in \overline{\mathrm{Conv}}_\mu(\mathbb{R}^n)\}
\end{aligned}$$

$\mathrm{Conv}(C)$ (resp., $\overline{\mathrm{Conv}}(C)$) is the set of convex (resp., closed convex) functions on $C$.

**Proposition**

*Let $\mu > 0$ and $C \neq \emptyset$ be a convex set. If $f \in \overline{\mathrm{Conv}}_{\mu}(C)$, then the problem*

$$f_* := \inf\{f(x) : x \in C\}$$

*has a unique optimal solution $x_*$ and*

$$f(x) \geq f_* + \frac{\mu}{2}\|x - x_*\|^2 \quad \forall x \in C.$$

### Proposition

*Assume that $f$ is differentiable on a convex set $\emptyset \neq C \subset \mathbb{R}^n$. Then, the following are equivalent:*

(a) *$f$ is convex on $C$, or equivalently, for every $x, x' \in C$ and $t \in (0, 1)$,*

$$f((1-t)x + tx') \leq (1-t)f(x) + tf(x')$$

(b) *for every $x', x \in C$,*

$$f(x') \geq f(x) + \langle \nabla f(x), x' - x \rangle$$

(c) *for every $x', x \in C$,*

$$\langle \nabla f(x') - \nabla f(x), x' - x \rangle \geq 0$$

## Proposition

*Assume that $f$ is differentiable on a convex set $\emptyset \neq C \subset \mathbb{R}^n$. Then, the following are equivalent:*

(a) *$f$ is striclty convex on $C$, or equivalently, for every $x, x' \in C$ such that $x \neq x'$ and $t \in (0, 1)$,*

$$f((1-t)x + tx') < (1-t)f(x) + tf(x')$$

(b) *for every $x', x \in C$ such that $x' \neq x$,*

$$f(x') > f(x) + \langle \nabla f(x), x' - x \rangle;$$

(c) *for every $x', x \in C$ such that $x' \neq x$,*

$$\langle \nabla f(x') - \nabla f(x), x' - x \rangle > 0$$

**Proposition**

*Assume that $f$ is differentiable on a convex set $\emptyset \neq C \subset \mathbb{R}^n$. Then, for any constant $\beta \in \mathbb{R}$, the following are equivalent:*

(a) $f - \beta\|\cdot\|^2/2$ *is convex on* $C$

(b) *for every* $x', x \in C$,

$$f(x') \geq f(x) + \langle \nabla f(x), x' - x \rangle + \frac{\beta}{2}\|x' - x\|^2 \quad (*)$$

(c) *for every* $x', x \in C$,

$$\langle \nabla f(x') - \nabla f(x), x' - x \rangle \geq \beta\|x' - x\|^2$$

Letting $\ell_f(x'; x) := f(x) + \langle \nabla f(x), x' - x \rangle$, then $(*)$ becomes

$$f(x') \geq \ell_f(x'; x) + \frac{\beta}{2}\|x' - x\|^2$$

**Def:** For $m > 0$, $f$ is $m$-**weakly convex on** $C$ if $f + m\|\cdot\|^2/2$ is convex on $C$. In such case, we say that $m$ is a **lower curvature** of $f$ on $C$

**Corollary**

*Assume that $f$ is differentiable on a convex set $\emptyset \neq C \subset \mathbb{R}^n$ and let $m > 0$ be given. Then, the following are equivalent:*

- *$f$ is $m$-weakly convex on $C$*
- *for all $x, x' \in C$,*

$$f(x') \geq \ell_f(x'; x) - \frac{m}{2}\|x' - x\|^2$$

- *for all $x, x' \in C$,*

$$\langle \nabla f(x') - \nabla f(x), x' - x \rangle \geq -m\|x' - x\|^2$$

**Proof:** Follows from previous prop with $\beta = -m$

## Definition

$M \in \mathbb{R}$ is an **upper curvature** of $f$ on $C$ if

$$\frac{M}{2}\|\cdot\|^2 - f(\cdot)$$

is convex on $C$

Clearly, $M \in \mathbb{R}$ is an upper curvature of $f$ on $C$ iff

$$f(x') \leq f(x) + \langle \nabla f(x), x' - x \rangle + \frac{M}{2}\|x' - x\|^2 \quad \forall x, x' \in C$$

or equivalently,

$$\langle \nabla f(x') - \nabla f(x), x' - x \rangle \leq M\|x' - x\|^2 \quad \forall x, x' \in C$$

## Definition

$f : \mathbb{R}^n \to (-\infty, \infty]$ is $L$-smooth on a nonempty convex set $C \subset \mathbb{R}^n$ if $f$ is differentiable on $C$ and $\nabla f$ is $L$-Lipschitz continuous on $C$, i.e.,

$$\|\nabla f(x') - \nabla f(x)\| \leq L\|x' - x\| \quad x, x' \in C$$

## Proposition

*Assume that $f$ is $L$-smooth on a convex set $\emptyset \neq C \subset \mathbb{R}^n$. Then, $L$ is both a lower and an upper curvature of $f$ on $C$, and hence $f$ is $L$-weakly convex on $C$*

Assume $f \in \mathrm{Conv}(\mathbb{R}^n)$ and $\bar{x} \in \mathrm{dom}\, f$

**Def:** $s \in \mathbb{R}^n$ is called a **subgradient** of $f$ at $\bar{x}$ if

$$f(x) \geq f(\bar{x}) + \langle s, x - \bar{x} \rangle \quad \forall x \in \mathbb{R}^n$$

The set of all subgradients of $f$ at $\bar{x}$ is denoted by $\partial f(\bar{x})$ and the point-to-set map $\partial f(\cdot)$ is called the **subdifferential** of $f$

### Proposition

*The following statements hold:*

a) *$\bar{x}$ is a global minimizer of*

$$\inf\{f(x) : x \in \mathbb{R}^n\}$$

*if and only if $0 \in \partial f(\bar{x})$*

b) *for every $x \in \mathrm{dom}\, f$, the set $\partial f(x)$ is (possibly, empty) closed convex*

Assume that $f : \mathbb{R}^n \to (-\infty, \infty]$ and $\bar{x} \in \operatorname{dom} f$

## Definition

The directional derivative of $f$ at $\bar{x}$ along $d \in \mathbb{R}^n$ is

$$f'(\bar{x}\,;\,d) := \lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t}$$

whenever the above limit exists (possibly, equal to $\pm\infty$).

## Proposition

*If $f$ is diffferentiable at $\bar{x}$ then, for every $d \in \mathbb{R}^n$, $f'(\bar{x}\,;\,d)$ exists and*

$$f'(\bar{x}\,;\,d) = f'(\bar{x})d = \langle \nabla f(\bar{x}), d \rangle$$

## Proposition

*Let* $f \in \mathrm{Conv}(\mathbb{R}^n)$, $\bar{x} \in \mathrm{dom}\, f$ *and* $d \in \mathbb{R}^n$ *be given. Then:*

a) *the function*

$$t > 0 \mapsto \frac{f(\bar{x} + td) - f(\bar{x})}{t}$$

*is non-decreasing*

b) *for every* $d \in \mathbb{R}^n$, $f'(\bar{x}\,;\, d)$ *is well-defined and*

$$f'(\bar{x}\,;\, d) = \inf_{t>0} \frac{f(\bar{x} + td) - f(\bar{x})}{t}$$

c) $f'(\bar{x}\,;\, \cdot)$ *is convex*

## Relationship of subgradient and directional derivative

**Proposition**

*Assume that* $f \in \mathrm{Conv}(\mathbb{R}^n)$ *and* $\bar{x} \in \mathrm{dom}\, f$. *Then,*

$$\partial f(\bar{x}) = \{s \in \mathbb{R}^n : \langle s, \cdot \rangle \leq f'(\bar{x}\,;\, \cdot)\}$$

*and*

$$\mathrm{cl}\,[f'(\bar{x}; \cdot)] = \sup\{\langle s, \cdot \rangle : s \in \partial f(\bar{x})\}$$

**Proposition**

*Assume that* $f \in \mathrm{Conv}(\mathbb{R}^n)$ *and* $\bar{x} \in \mathrm{dom}\, f$. *Then:*

a) $\partial f(\bar{x}) = \emptyset$ *if and only if there exists* $d_0 \in \mathbb{R}^n$ *such that*

$$f'(\bar{x}; d_0) = -\infty$$

b) *if* $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}\, f)$, *then* $\partial f(\bar{x}) \neq \emptyset$ *and*

$$f'(\bar{x}; d) = \sup\{\langle\, d, s \,\rangle : s \in \partial f(\bar{x})\}$$

c) $\bar{x} \in \mathrm{int}\,(\mathrm{dom}\, f)$ *if and only if* $\partial f(\bar{x})$ *is non-empty and bounded, in which case*

$$f'(\bar{x}; d) = \max\{\langle\, d, s \,\rangle : s \in \partial f(\bar{x})\}$$

# Subgradient versus differentiability

**Proposition**

*Assume that* $f \in \mathrm{Conv}(\mathbb{R}^n)$ *and* $\bar{x} \in \mathrm{dom}\, f$*. Then* $\partial f(\bar{x})$ *is a singleton if and only if* $f$ *is diferentiable at* $\bar{x}$*, in which case*

$$\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$$

**Def:** The **conjugate** of a function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is

$$f^*(s) := \sup_{x \in \mathbb{R}^n} \langle s, x \rangle - f(x), \quad \forall s \in \mathbb{R}^n$$

**Proposition**

*Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be given. Then, the following statements hold:*

a) $f = +\infty$ *if and only if* $f^* = -\infty$

b) *if* $f(x^0) = -\infty$ *for some* $x^0 \in \mathbb{R}^n$, *then* $f^* = +\infty$

c) *if* $f \neq \infty$ *and* $f$ *is minorized by some affine function (e.g.,* $f \in \mathrm{Conv}(\mathbb{R}^n)$*), then* $f^* \in \overline{\mathrm{Conv}}(\mathbb{R}^n)$

## Proposition

*Let* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ *be given. Then:*

a) $-f^*(0) = \inf\{f(x) : x \in \mathbb{R}^n\}$

b) *for every* $s \in \mathbb{R}^n$,

$$f^*(s) = -\inf\{f(x) - \langle s, x\rangle : x \in \mathbb{R}^n\} \\ = \inf\{\beta \in \mathbb{R} : f \geq \langle s, \cdot \rangle - \beta\}$$

*and the infimum is achieved whenever* $f^*(s)$ *is finite*

c) (*Fenchel's inequality*) *for any* $x, s \in \mathbb{R}^n$,

$$f^*(s) \geq \langle x, s \rangle - f(x)$$

**Proposition**

*Let* $f \in \mathrm{Conv}(\mathbb{R}^n)$ *and* $x \in \mathrm{dom}\, f$ *be given. Then,*

$$s \in \partial f(x) \iff f^*(s) \leq \langle\, s, x \,\rangle - f(x)$$

*(or equivalently,* $f^*(s) = \langle\, s, x \,\rangle - f(x)$*)*

**Proposition**

*For functions $f, g : \mathbb{R}^n \to \bar{\mathbb{R}}$, vectors $x_0, s_0 \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$, the following statements hold:*

a) *if $g = f + \alpha$, then $g^* = f^* - \alpha$*
b) *if $\alpha > 0$ and $g = \alpha f$, then $g^*(s) = \alpha f^*(s/\alpha)$ for every $s \in \mathbb{R}^n$*
c) *if $\alpha \neq 0$ and $g(x) = f(\alpha x)$ for every $x \in \mathbb{R}^n$, then $g^*(s) = f^*(s/\alpha)$ for every $s \in \mathbb{R}^n$*
d) *if $g(x) = f(x - x_0)$ for every $x \in \mathbb{R}^n$, then $g^*(s) = f^*(s) + \langle s, x_0 \rangle$ for every $s \in \mathbb{R}^n$*
e) *if $g(x) = f(x) + \langle x, s_0 \rangle$ for every $x \in \mathbb{R}^n$, then $g^*(s) = f^*(s - s_0)$ for every $s \in \mathbb{R}^n$*
f) *if $f \leq g$, then $f^* \geq g^*$*

**Proposition**

*If* $f \in \overline{\mathrm{Conv}}\left(\mathbb{R}^n\right)$ *then*

$$f^{**} = f$$

*where* $f^{**} := \left(f^*\right)^*$

*As a consequence, for every* $x \in \mathbb{R}^n$,

$$f(x) = \sup_s \langle s, x \rangle - f^*(s)$$

## Proposition

*Assume that* $f \in \overline{\mathrm{Conv}}\,(\mathbb{R}^n)$. *Then,*

$$s \in \partial f(x) \iff x \in \partial f^*(s)$$

*As a consequence,* $\partial f^*(0)$ *is equal to the set of minimizers of*

$$\inf\{f(x) : x \in \mathbb{R}^n\}$$

**Proposition**

*Assume that $f \in \overline{\mathrm{Conv}}\left(\mathbb{R}^n\right)$. Then, the following are equivalent:*

a) *$f$ is $\mu$-convex*

b) *$f^*$ is $(1/\mu)$-smooth on $\mathbb{R}^n$*

**Application:** Assume that $h \in \overline{\mathrm{Conv}}_\mu\left(\mathbb{R}^n\right)$ and $D$ is a closed convex set such that $D \cap \operatorname{dom} h \neq \emptyset$. Define

$$g(x) = \max_{y \in D} \langle y, Ax \rangle - h(y) \quad \forall x \in \mathbb{R}^n$$

Then, $g$ is $(\|A\|^2/\mu)$-smooth

Let $f \in \mathrm{Conv}(\mathbb{R}^n)$, $\bar{x} \in \mathrm{dom}\, f$ and $\varepsilon \geq 0$ be given

**Definition**

$s \in \mathbb{R}^n$ is called an $\varepsilon$-**subgradient** of $f$ at $\bar{x}$ if

$$f(x) \geq f(\bar{x}) + \langle s, x - \bar{x} \rangle - \varepsilon \quad \forall x \in \mathbb{R}^n.$$

The set of all $\varepsilon$-subgradients of $f$ at $\bar{x}$ is denoted by $\partial_\varepsilon f(\bar{x})$ and the point-to-set map $\partial_\varepsilon f(\cdot)$ is called the $\varepsilon$-**subdifferential** of $f$

**Proposition**

*The following statements hold:*

a) $\bar{x}$ *is an* $\varepsilon$*-minimizer of* $f_* := \inf\{f(x) : x \in \mathbb{R}^n\}$, *i.e.,*

$$f(\bar{x}) - f_* \leq \varepsilon$$

*or equivalently,*

$$f(\bar{x}) \leq f(x) + \varepsilon \quad \forall x \in \mathbb{R}^n$$

*if and only if* $0 \in \partial_\varepsilon f(\bar{x})$

b) $\partial_\varepsilon f(\bar{x})$ *is a (possibly, empty) closed convex set*

## Relationship of $\varepsilon$-subgradients and conjugate function

**Proposition**

*Let* $f \in \mathrm{Conv}(\mathbb{R}^n)$, $x \in \mathrm{dom}\, f$ *and* $s \in \mathbb{R}^n$ *be given. Then,*

$$s \in \partial_\varepsilon f(x) \iff f^*(s) \leq \langle s, x \rangle - f(x) + \varepsilon$$

## Transportation formula for subgradient

**Proposition**

*Let* $f \in \mathrm{Conv}(\mathbb{R}^n)$, $x \in \mathrm{dom}\, f$ *and* $s \in \mathbb{R}^n$ *be given and define*

$$\bar{\varepsilon} := f^*(s) + f(x) - \langle s, x \rangle$$

*Then, the following statements hold:*

(a) *if* $\bar{\varepsilon} < \infty$, *then*

$$\bar{\varepsilon} = \min\{\varepsilon : s \in \partial_\varepsilon f(x)\}$$

(b) *if* $s \in \partial f(\bar{x})$ *for some* $\bar{x} \in \mathrm{dom}\, f$, *then* $s \in \partial_{\bar{\varepsilon}} f(x)$ *and*

$$\bar{\varepsilon} = f(x) - f(\bar{x}) - \langle s, x - \bar{x} \rangle < \infty$$

**Proposition**

*If* $f \in \mathrm{Conv}(\mathbb{R}^n)$ *and* $A : \mathbb{R}^m \to \mathbb{R}^n$ *is a linear map, then* $f \circ A$ *is convex and*

$$\partial_\varepsilon (f \circ A)(x) \supset A^* \partial_\varepsilon f(Ax) \quad \forall x \in \mathbb{R}^m$$

*and equality holds whenever*

$$A(\mathbb{R}^m) \cap \mathrm{ri}\,(\mathrm{dom}\, f) \neq \emptyset$$

**Proposition**

*If* $f_i \in \overline{\mathrm{Conv}}\,(\mathbb{R}^n)$ *for* $i = 1, \ldots, m$*, then* $f := f_1 + \ldots + f_m$ *is convex and*

$$\partial f(x) \supset \partial f_1(x) + \ldots + \partial f_m(x) \quad \forall x \in \mathbb{R}^n$$

*and equality holds whenever*

$$\cap_{i=1}^{m} \mathrm{ri}\,(\mathrm{dom}\, f_i) \neq \emptyset$$