[section]

ISyE8813
Alternating direction method of multipliers (ADMM)

Renato D.C. Monteiro

March 27, 2024

Consider the problem

$$\begin{aligned} &\min\ f(x) + g(x) \\ &\text{s.t. } Ax + By = b \in \mathbb{R}^r \end{aligned}$$

where $f \in \overline{\mathrm{Conv}}\,(\mathbb{R}^n)$ and $g \in \overline{\mathrm{Conv}}\,(\mathbb{R}^m)$. For a given $\rho > 0$, define

$$L_\rho(x, y; p) = f(x) + g(x) + p^T(Ax + By - b) + \frac{\rho}{2}\|Ax + By - b\|^2$$

1 Augmented Lagrangian Method

Algorithm 1 (ALM Method)

0. Let $p_0 \in \mathbb{R}^r$ be given.
1. Compute
$$(x, y) \in \mathrm{Argmin}_{(x', y')} L_\rho(x', y'; p_0). \tag{1}$$
2. Set
$$p = p_0 + \rho(Ax + By - b)$$
3. Set $p_0 \leftarrow p$ and go to step 1.

Optimality Conditions for (1)

$$\begin{aligned} 0 &\in \partial f(x) + A^*[p_0 + \rho(Ax + By - b)] \\ 0 &\in \partial g(y) + B^*[p_0 + \rho(Ax + By - b)] \\ \frac{p_0 - p}{\rho} &= b - Ax - By \end{aligned}$$

or equivalently,

$$\begin{aligned} 0 &\in \partial f(x) + A^* p \\ 0 &\in \partial g(y) + B^* p \\ \frac{p_0 - p}{\rho} &= b - Ax - By \end{aligned} \tag{2}$$

We will now see that ALM is an exact implementation of PPM for a certain Bregman distance. Indeed, for any $z = (x, y; p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r$, define

$$T(z) = T(x, y; p) := \begin{bmatrix} 0 & 0 & A^* \\ 0 & 0 & B^* \\ -A & -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ p \end{bmatrix} + \begin{bmatrix} \partial f(x) \\ \partial g(y) \\ b \end{bmatrix}.$$

and the distance generating function

$$w(z) = w(x, y; p) := \frac{1}{2}\|p\|^2$$

Then, it can be shown that T is maximal monotone and that

$$(dw)_{z_0}(z) = \frac{1}{2}\|p - p_0\|^2 \tag{3}$$

and

$$\nabla w(z) = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

Also, (18) becomes

$$\frac{\nabla w(z_0) - \nabla w(z)}{\rho} \in T(z)$$

This is the equation of an exact PPM with respect to the Bregman distance (3) and prox stepsize λ equal to ρ

2 Generalized HPE framework

Consider the MIP

$$0 \in T(z)$$

where $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone.

Assume that for some semi-norm in $\mathbb{R}^n$, the following conditions hold:

1) $T^{-1}(0) \neq \emptyset$

2) there exists $m, M > 0$ such that for every $z, z' \in \mathbb{R}^n$, have

$$(dw)_z(z') \geq \frac{m}{2}\|z' - z\|^2$$
$$\|\nabla w(z') - \nabla w(z)\|^* \leq M \|z' - z\|$$

where

$$\|\cdot\|^* = \sup\{\langle \cdot, v\rangle : \|v\| \leq 1\}$$

Condition 2) implies that

$$\frac{m}{2}\|z - z'\|^2 \leq (dw)_z(z') \leq \frac{M}{2}\|z - z'\|^2 \quad \forall z, z' \in Z. \tag{4}$$

Example: If

$$w(\cdot) = (1/2)\|\cdot\|_Q^2$$

where Q is a self-adjoint positive semidefinite linear operator, then w satisfies condition 2 with $(m, M) = (1, 1)$.

Proposition 2.1. *If $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a self-adjoint positive semidefinite linear operator, then the semi-norm*

$$\|\cdot\| := \langle \cdot, Q(\cdot)\rangle^{1/2}$$

satisfies the following statements:

(a) $\operatorname{dom}\|\cdot\|^* = \operatorname{Im}(Q)$ *and*

$$\|Qz\|^* = \|z\| \quad \forall z \in \mathbb{R}^n$$

(b) if Q is invertible, then

$$\|z\|^* = \langle Q^{-1}z, z\rangle^{1/2} \quad \forall z \in \mathbb{R}^n$$

Algorithm 2 (Inexact PPM framework with Bregman distance)

0. Let $z_0 \in \mathbb{R}^n$ and $\sigma \in [0,1]$ be given.
1. Find $\lambda > 0$ and $(z, \tilde{z}, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ such that

$$\frac{\nabla w(z_0) - \nabla w(z)}{\lambda} \in T^{\varepsilon}(\tilde{z}) \tag{5}$$

$$(dw)_z(\tilde{z}) + \lambda\varepsilon \le \sigma (dw)_{z_0}(\tilde{z}) \tag{6}$$

2. Set $z_0 \leftarrow z$ and go to step 1.

Hence, sequence-wise, we have for every $k \ge 1$ that

$$r_k := \frac{\nabla w(z_{k-1}) - \nabla w(z_k)}{\lambda_k} \in T^{\varepsilon_k}(\tilde{z}_k) \tag{7}$$

$$(dw)_z(\tilde{z}_k) + \lambda_k \varepsilon_k \le \sigma (dw)_{z_{k-1}}(\tilde{z}_k) \tag{8}$$

Proposition 2.2. *(***Pointwise)** *Assume that $\sigma < 1$ and $\lambda_k \ge \underline{\lambda}$ for every $k \ge 1$. Then, for every $k \ge 1$, there exists $i \le k$ such that*

$$\|r_i\|^* \le \frac{2M}{\underline{\lambda}\sqrt{mk}} \sqrt{\frac{(1+\sigma)(dw)_0}{1-\sigma}} = \mathcal{O}\left(\frac{1}{\underline{\lambda}\sqrt{k}}\right)$$

and

$$\varepsilon_i \le \frac{(1+\sigma)(dw)_0}{(1-\sigma)\underline{\lambda}k} = \mathcal{O}\left(\frac{1}{\underline{\lambda}k}\right)$$

where r_i is as in (7) *and*

$$(dw)_0 := \inf\left\{(dw)_{z_0}(z^*) : z^* \in T^{-1}(0)\right\}$$

For $k \geq 1$, define $\Lambda_k := \sum_{i=1}^k \lambda_i$ and the ergodic iterate $(\tilde{z}_k^a, r_k^a, \varepsilon_k^a)$ as

$$\tilde{z}_k^a = \frac{1}{\Lambda_k} \sum_{i=1}^k \lambda_i \tilde{z}_i, \quad r_k^a := \frac{1}{\Lambda_k} \sum_{i=1}^k \lambda_i r_i, \quad \varepsilon_k^a := \frac{1}{\Lambda_k} \sum_{i=1}^k \lambda_i \left(\varepsilon_i + \langle r_i, \tilde{z}_i - \tilde{z}_k^a \rangle\right). \tag{9}$$

Theorem 2.3. (Ergodic convergence of the NE-HPE) *For every $k \geq 1$, we have*

$$r_k^a \in T^{\varepsilon_k^a}(\tilde{z}_k^a)$$

and

$$\|r_k^a\|^* \leq \frac{2\sqrt{2}M(dw_0)^{1/2}}{\sqrt{m}\Lambda_k}, \quad \varepsilon_k^a \leq \left(\frac{3M}{m}\right)\left[\frac{3(dw)_0 + \sigma\theta_k}{\Lambda_k}\right].$$

where

$$\theta_k := \max_{i=1,\ldots,k} (dw)_{z_{i-1}}(\tilde{z}_i). \tag{10}$$

Moreover, the sequence $\{\theta_k\}$ is bounded under either one of the following situations:

(a) $\sigma < 1$, *in which case*

$$\theta_k \leq \frac{(dw)_0}{1-\sigma}; \tag{11}$$

(b) $\mathrm{Dom}\, T$ *is bounded, in which case*

$$\theta_k \leq \frac{2M}{m}[(dw)_0 + D]$$

where

$$D := \sup\left\{\min\{(dw)_y(y'), (dw)_{y'}(y)\} : y, y' \in \mathrm{Dom}\, T\right\}$$

3 ADDM

Algorithm 3 (ADMM)

0. Let $(y_0, p_0) \in \mathbb{R}^m \times \mathbb{R}^r$ be given.
1. Compute
$$x \in \operatorname{Argmin}_{x'} L_\rho(x', y_0; p_0). \tag{12}$$
2. Compute
$$y \in \operatorname{Argmin}_{y'} L_\rho(x, y'; p_0). \tag{13}$$
3. Set
$$p = p_0 + \rho(Ax + By - b)$$
4. Set $(y_0, p_0) \leftarrow (y, p)$ and go to step 1.

Optimality Conditions for (17) **and** (18)

$$\begin{aligned} 0 &\in \partial f(x) + A^*[p_0 + \rho(Ax + By_0 - b)) \\ 0 &\in \partial g(y) + B^*[p_0 + \rho(Ax + By - b)] \\ \frac{p_0 - p}{\rho} &= b - Ax - By \end{aligned}$$

or equivalently,

$$\begin{aligned} 0 &\in \partial f(x) + A^*\tilde{p} \\ 0 &\in \partial g(y) + B^* p \\ \frac{p_0 - p}{\rho} &= b - Ax - By \end{aligned} \tag{14}$$

where

$$\tilde{p} = p_0 + \rho(Ax + By_0 - b)$$

Hence,

$$p - \tilde{p} = \rho B(y - y_0) \tag{15}$$

and so the second inclusion in (14) implies that

$$\begin{aligned} \rho B^* B(y_0 - y) &\in \partial g(y) + B^* p + \rho B^* B(y_0 - y) \\ &= \partial g(y) + B^*[p + B(y_0 - y)] = \partial g(y) + B^*\tilde{p} \end{aligned}$$

Thus, (14) is equivalent to

$$\begin{aligned} 0 &\in \partial f(x) + A^*\tilde{p} \\ \rho B^* B(y_0 - y) &\in \partial g(y) + B^*\tilde{p} \\ \frac{p_0 - p}{\rho} &= b - Ax - By \end{aligned} \tag{16}$$

Defining

$$\tilde{z} := (x, y; \tilde{p}), \quad w(z) := \frac{1}{2\rho}\|p\|^2 + \frac{\rho}{2}\|By\|^2$$

and noting that

$$\nabla w(z) = \begin{bmatrix} 0 \\ \rho B^* By \\ \rho^{-1} p \end{bmatrix}$$

it follows that (16) is equivalent to

$$\nabla w(z_0) - \nabla w(z) \in T(\tilde{z})$$

which shows that (5) is satisfied with $\varepsilon = 0$ and $\lambda = 1$.

We will now show that (6) is also satisfied with $\varepsilon = 0$, $\lambda = 1$ and $\sigma = 1$. Indeed, since $\varepsilon = 0$, the left hand side of (6) becomes

$$(dw)_z(\tilde{z}) + \lambda\varepsilon = (dw)_z(\tilde{z}) = \frac{1}{2\rho}\|\tilde{p} - p\|^2 = \frac{\rho}{2}\|B(y - y_0)\|^2$$

where the last equality is due to (15). On the other hand, since $\sigma = 1$, the right hand side of (6) becomes

$$\sigma(dw)_{z_0}(\tilde{z}) = (dw)_{z_0}(\tilde{z}) = \frac{1}{2\rho}\|\tilde{p} - p_0\|^2 + \frac{\rho}{2}\|B(y - y_0)\|^2$$

Thus, (6) holds with $\varepsilon = 0$, $\lambda = 1$ and $\sigma = 1$.

Is the quantity θ_k in (10) bounded? Since $\sigma = 1$, this fact is not clear. Note that

$$\begin{aligned}
(dw)_{z_0}(\tilde{z}) &= \frac{1}{2\rho}\|\tilde{p} - p_0\|^2 + \frac{\rho}{2}\|B(y - y_0)\|^2 \\
&\leq \frac{1}{\rho}\left[\|\tilde{p} - p\|^2 + \|p - p_0\|^2\right] + \frac{\rho}{2}\|B(y - y_0)\|^2 \\
&= \frac{1}{\rho}\left[\rho^2\|B(y - y_0)\|^2 + \|p - p_0\|^2\right] + \frac{\rho}{2}\|B(y - y_0)\|^2 \\
&= \frac{1}{\rho}\|p - p_0\|^2 + \frac{3\rho}{2}\|B(y - y_0)\|^2 \leq 3(dw)_{z_0}(z) \\
&\leq 6\left[(dw)_{z_0}(z^*) + (dw)_z(z^*)\right]
\end{aligned}$$

Since the quantity $(dw)_z(z^*)$ remains bounded (why?), it follows that $(dw)_{z_0}(\tilde{z})$ also remains bounded, and so does the quantity θ_k in (10)

4 Proximal ADMM

Algorithm 4 (Proximal ADMM)

0. Let $\theta > 0$, $(y_0, p_0) \in \mathbb{R}^m \times \mathbb{R}^r$, and positive-semidefinite self-adjoint linear operators $\mathcal{G}$ and $\mathcal{H}$ be given.
1. Compute
$$x \in \operatorname{Argmin}_{x'} L_\rho(x', y_0; p_0) + \frac{1}{2}\|x - x_{k-1}\|^2_{\mathcal{G}}. \tag{17}$$
2. Compute
$$y \in \operatorname{Argmin}_{y'} L_\rho(x, y'; p_0) + \frac{1}{2}\|y - y_{k-1}\|^2_{\mathcal{H}} \tag{18}$$
3. Set
$$p = p_0 + \theta\rho(Ax + By - b)$$
4. Set $(y_0, p_0) \leftarrow (y, p)$ and go to step 1.

Theorem 4.1. (Pointwise convergence of the proximal ADMM) *Consider the sequence $\{(x_k, y_k, p_k)\}$ generated by the proximal ADMM with $\theta \in (0, (\sqrt{5} + 1)/2)$, and let $\{\tilde{x}_k\}$ be defined as*

$$\tilde{p}_k = p_{k-1} + \rho(Ax_k + By_{k-1} - b). \tag{19}$$

Then, for every $k \in \mathbb{N}$,

$$\begin{pmatrix} G(x_{k-1} - x_k) \\ (H + \rho B^* B)(y_{k-1} - y_k) \\ (p_{k-1} - p_k)/(\rho\theta) \end{pmatrix} \in \begin{bmatrix} \partial f(x_k) + A^* \tilde{p}_k \\ \partial g(y_k) + B^* \tilde{p}_k \\ Ax_k + By_k - b \end{bmatrix} \tag{20}$$

and there exists $i \leq k$ such that

$$\left(\|x_{i-1} - x_i\|^2_{\mathcal{G}} + \|y_{i-1} - y_i\|^2_{(\mathcal{H} + \rho B^* B)} + \frac{1}{\beta\theta} \|p_{i-1} - p_i\|^2_{\mathcal{X}} \right) = \mathcal{O}\left(\frac{1}{k}\right)$$

5 Chambolle and Poch algorithm

Consider the problem

$$(P) \quad \min_x \max_y \langle Kx, y\rangle + G(x) - F^*(y)$$

where $G \in \overline{\text{Conv}}\,(\mathbb{R}^n)$, $F \in \overline{\text{Conv}}\,(\mathbb{R}^m)$, $K : \mathbb{R}^n \mapsto \mathbb{R}^m$ is linear. The problem (P) is equivalent to

$$\min_x \; F(Kx) + G(x)$$

and has the dual formulation

$$\max_x \min_y \langle Kx, y\rangle + G(x) - F^*(y) =: \psi(x, y)$$

or equivalently

$$\max_y -G^*(-K^*y) - F^*(y).$$

Furthermore, assume that there exists $(x^*, y^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$-Kx^* + \partial F^*(y^*) \ni 0, \quad K^*y^* + \partial G(x^*) \ni 0$$

or equivalently

$$(0, 0) \in \partial\left[\psi(\cdot, y^*) - \psi(x^*, \cdot)\right](x^*, y^*)$$

Algorithm 5 (Chambolle-Poch (CP) Algorithm)

---

0. Choose $\tau_1, \tau_2 > 0$, $\theta = 1$, $(p_0, q_0) \in \mathbb{R}^n \times \mathbb{R}^m$ and set $\bar{p}_0 = p_0$ and $k = 1$
1. compute

$$\begin{aligned}
q_k &= (I + \tau_2 \partial F^*)^{-1}(q_{k-1} + \tau_2 K \bar{p}_{k-1}) \\
p_k &= (I + \tau_1 \partial G)^{-1}(p_{k-1} - \tau_1 K^* q_k) \\
\bar{p}_k &= p_k + \theta(p_k - p_{k-1})
\end{aligned}$$

2. set $k \leftarrow k + 1$ and go to step 1.

We have

$$\frac{q_k - q_{k-1}}{\tau_2} - K\bar{p}_{k-1} + \partial F^*(q_k) \ni 0 \tag{21}$$

$$\frac{p_k - p_{k-1}}{\tau_1} + K^*q_k + \partial G(p_k) \ni 0 \tag{22}$$

Proposition 5.1. *The CP algorithm is an instance of the inexact IPP framework with $\sigma^2 := \|K\|^2\tau_1\tau_2$.*

Fix $k \geq 1$ and let

$$\tilde{x}_k = p_k, \quad x_k = \theta p_k, \quad \tilde{y}_k = q_k$$

Then, (22) implies that

$$\frac{x_{k-1} - x_k}{\theta\tau_1} \in K^*\tilde{y}_k + \partial G(\tilde{x}_k)$$

and (21) implies that

$$\begin{aligned}
0 &\in \frac{q_k - q_{k-1}}{\tau_2} - K\bar{p}_{k-1} + \partial F^*(q_k) \\
&= \frac{q_k - q_{k-1}}{\tau_2} + K(\bar{p}_k - \bar{p}_{k-1}) + K(p_k - \bar{p}_k) - Kp_k + \partial F^*(q_k) \\
&= \frac{q_k - q_{k-1}}{\tau_2} + K(\bar{p}_k - \bar{p}_{k-1}) - \theta K(p_k - p_{k-1}) - Kp_k + \partial F^*(q_k) \\
&= \frac{y_k - y_{k-1}}{\tau_2} - K\tilde{x}_k + \partial F^*(\tilde{y}_k)
\end{aligned}$$

where

$$y_k := q_k + \tau_2 K[\bar{p}_k - \theta p_k]$$

Now, consider the distance generating function

$$w(x, y) = \frac{1}{\tau_1}\|x\|^2 + \frac{1}{\tau_2}\|y\|^2$$

Then, if $z = (x_k, y_k)$, $\tilde{z} = (\tilde{x}_k, \tilde{y}_k)$ and $z_0 = (x_{k-1}, y_{k-1})$, then

$$\begin{aligned}
(dw)_z(\tilde{z}) &= \frac{1}{\tau_1}\|x_k - \tilde{x}_k\|^2 + \frac{1}{\tau_2}\|y_k - \tilde{y}_k\|^2 = \frac{1}{\tau_1}\|\theta p_k - p_k\|^2 + \frac{1}{\tau_2}\|y_k - q_k\|^2 \\
&= \frac{(1-\theta)^2}{\tau_1}\|p_k\|^2 + \frac{1}{\tau_2}\|\tau_2 K(\bar{p}_k - \theta p_k)\|^2 = \frac{(1-\theta)^2}{\tau_1}\|p_k\|^2 + \tau_2\|K\|^2\|\bar{p}_k - \theta p_k\|^2 \\
&= \frac{(1-\theta)^2}{\tau_1}\|p_k\|^2 + \tau_2\|K\|^2\|p_k - \theta p_{k-1}\|^2
\end{aligned}$$

Also,

$$\begin{aligned}
(dw)_{z_0}(\tilde{z}) &= \frac{1}{\tau_1}\|x_{k-1} - \tilde{x}_k\|^2 + \frac{1}{\tau_2}\|y_{k-1} - \tilde{y}_k\|^2 \\
&\geq \frac{1}{\tau_1}\|x_{k-1} - \tilde{x}_k\|^2 = \frac{1}{\tau_1}\|\theta p_{k-1} - p_k\|^2
\end{aligned}$$

6 A block-decomposition algorithm

Algorithm 6 (Block-Decomposition Algorithm)

0. Choose $\tau_1, \tau_2 > 0$, $(p_0, q_0) \in \mathbb{R}^n \times \mathbb{R}^m$ and set $k = 1$
1. Compute

$$\begin{aligned}
\tilde{q}_k &= (I + \tau_2 \partial F^*)^{-1}(q_{k-1} + \tau_2 K p_{k-1}) \\
p_k &= (I + \tau_1 \partial G)^{-1}(p_{k-1} - \tau_1 K^* \tilde{q}_k) \\
q_k &= \tilde{q}_k + \tau_2 K[p_k - p_{k-1}]
\end{aligned}$$

2. Set $k \leftarrow k + 1$ and go to step 1

Have

$$\frac{\tilde{q}_k - q_{k-1}}{\tau_2} - K p_{k-1} + \partial F^*(\tilde{q}_k) \ni 0 \tag{23}$$

$$\frac{p_k - p_{k-1}}{\tau_1} + K^* \tilde{q}_k + \partial G(p_k) \ni 0 \tag{24}$$

Fix $k \geq 1$ and let

$$\tilde{x}_k = x_k = p_k, \quad \tilde{y}_k = \tilde{q}_k, \quad y_k = q_k$$

Then, (24) implies that

$$\frac{x_{k-1} - x_k}{\tau_1} \in K^* \tilde{y}_k + \partial G(\tilde{x}_k)$$

and (23) implies that

$$\begin{aligned}
0 &\in \frac{\tilde{q}_k - q_{k-1}}{\tau_2} - K p_{k-1} + \partial F^*(\tilde{q}_k) \\
&= \frac{\tilde{q}_k - q_{k-1}}{\tau_2} + K(p_k - p_{k-1}) - K p_k + \partial F^*(\tilde{y}_k) \\
&= \frac{\tilde{q}_k - q_{k-1}}{\tau_2} + \frac{q_k - \tilde{q}_k}{\tau_2} - K p_k + \partial F^*(\tilde{y}_k) \\
&= \frac{q_k - q_{k-1}}{\tau_2} - K p_k + \partial F^*(\tilde{y}_k) \\
&= \frac{y_k - y_{k-1}}{\tau_2} - K \tilde{x}_k + \partial F^*(\tilde{y}_k)
\end{aligned}$$

and hence that

$$\frac{y_{k-1} - y_k}{\tau_2} \in -K \tilde{x}_k + \partial F^*(\tilde{y}_k)$$

Now, consider the distance generating function

$$w(x, y) = \frac{1}{\tau_1}\|x\|^2 + \frac{1}{\tau_2}\|y\|^2$$

Then, if $z = (x_k, y_k)$, $\tilde{z} = (\tilde{x}_k, \tilde{y}_k)$ and $z_0 = (x_{k-1}, y_{k-1})$, then

$$\begin{aligned}
(dw)_z(\tilde{z}) &= \frac{1}{\tau_1}\|x_k - \tilde{x}_k\|^2 + \frac{1}{\tau_2}\|y_k - \tilde{y}_k\|^2 = \frac{1}{\tau_2}\|q_k - \tilde{q}_k\|^2 \\
&= \frac{1}{\tau_2}\|\tau_2 K(p_k - p_{k-1})\|^2 \leq \tau_2\|K\|^2\|p_k - p_{k-1}\|^2
\end{aligned}$$

Also,

$$\begin{aligned}
(dw)_{z_0}(\tilde{z}) &= \frac{1}{\tau_1}\|x_{k-1} - \tilde{x}_k\|^2 + \frac{1}{\tau_2}\|y_{k-1} - \tilde{y}_k\|^2 \\
&\geq \frac{1}{\tau_1}\|x_{k-1} - \tilde{x}_k\|^2 = \frac{1}{\tau_1}\|p_{k-1} - p_k\|^2
\end{aligned}$$

Hence, if

$$\sigma := \tau_1\tau_2\|K\|^2 \leq 1$$

then

$$(dw)_z(\tilde{z}) \leq \tau_2\|K\|^2\|p_k - p_{k-1}\|^2 = \frac{\sigma}{\tau_1}\|p_k - p_{k-1}\|^2 \leq \sigma(dw)_{z_0}(\tilde{z})$$

Hence, the two conditions for the IPP framework holds with $\varepsilon = 0$ and $\lambda = 1$